M.Sc. II Power Series Paper VI

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(b) Suppose $|z| > R$, and choose r such that $|z| > r > R$, $\left(\frac{1}{r} < \frac{1}{R}\right)$. Thus by definition of the limit sup, there are infinitely many integers n such that $|a_n|^{1/n} > \frac{1}{r}$. It follows that $|a_n z^n| > \left(\frac{|z|}{r}\right)^n$ for all $n \geq N$ and, since $\left(\frac{|z|}{r}\right) > 1$ these terms becomes unbounded and so the series diverges.

(c) If $0 < r < R$, choose ρ such that $r < \rho < R$. As in (a) we have $|a_n| < \frac{1}{\rho^n}$ for all $n \geq N$.

Thus if $|z| \leq r$, $|a_n z^n| \leq \left(\frac{|z|}{\rho}\right)^n \leq \left(\frac{r}{\rho}\right)^n$ and $\left(\frac{r}{\rho}\right) < 1$.

Hence, by Weierstrass M-test, the power series $\sum_{n=0}^{\infty} a_n z^n$ converges uniformely on $|z| \leq r$.

Definition:- The circle $|z-a| = R$ which includes in its interior $|z-a| < R$, in which the power series $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges, is called circle of convergence. Radius R of this circle is called radius of convergence of power series and in view of above result it is given by $\frac{1}{R} = \lim\sup |a_n|^{1/n}$.

Ex. Find the radius of convergence for the series

$$\sum_{n=0}^{\infty} \frac{z^n}{n^n}$$

Soln. Here $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$ and $a_n = \frac{1}{n^n}$, $a = 0$.

Therefore $\frac{1}{R} = \text{Lt sup} |a_n|^{\frac{1}{n}} = \text{Lt sup} \left|\frac{1}{n^n}\right|^{\frac{1}{n}} = \text{Lt sup} \frac{1}{n} = 0$

Thus radius of convergence for the series $\sum_{n=0}^{\infty} \frac{z^n}{n^n}$ is $R = \infty$.

That is the series converges in whole complex plane.

<u>Theorem</u>:- If $\sum_{n=0}^{\infty} a_n (z-a)^n$ is given power series with radius of convergence R, then $R = \lim \left|\frac{a_n}{a_{n+1}}\right|$ if this limit exists.

Proof:- we assume that $a=0$ and let $\alpha = \lim \left|\frac{a_n}{a_{n+1}}\right|$.
Suppose that $|z| < r < \alpha$ and find an integer N such that $r < \left|\frac{a_n}{a_{n+1}}\right|$ for all $n \geq N$. Let $B = |a_N| r^N$ then

$$|a_{N+1}| r^{N+1} = |a_{N+1}| r r^N < |a_N| r^N = B,$$

$$|a_{N+2}|^{N+2} = |a_{N+2}| r r^{N+1} < |a_{N+1}| r^{N+1} < B.$$

Continuining this we get, $|a_n r^n| \leq B$ for all $n \geq N$.
Then $|a_n z^n| = |a_n r^n| \frac{|z|^n}{r^n} \leq B \frac{|z|^n}{r^n}$ for all $n \geq N$.
Since $|z| < r$, we get that $\sum_{n=0}^{\infty} |a_n z^n|$ is dominated by convergent series and hence converges.
Since $r < \alpha$ was arbitrary this gives that $\alpha \leq R$.
Now if $|z| > r > \alpha$ then $|a_n| < r|a_{n+1}|$ for all $n \geq N$.
As above we get $|a_n r^n| \geq B = |a_N r^N|$ for all $n \geq N$.
This gives that $|a_n z^n| \geq B \frac{|z|^n}{|r|^n} \to \infty$ as $n \to \infty$.
Hence $\sum_{n=0}^{\infty} a_n z^n$ diverges and so $R \leq \alpha$. Thus $R = \alpha$.

<u>Ex.</u> Find radius of convergence for the series $\sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$

<u>Soln.</u> Here $\sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$ and $a_n = \frac{1}{\lfloor n}$, $a = 0$.

Therefore $\frac{1}{R} = \lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{\lfloor n}{\lfloor (n+1)}\right| = \lim \left|\frac{1}{(n+1)}\right| = 0$

This radius of convergence for the series $\sum_{n=0}^{\infty} \frac{z^n}{\lfloor n}$ is $R = \infty$.